State Feedback Control Synthesis

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Stabilizing Controller

Stabilizing Controller

- Given system $\dot{x} = Ax + Bu$
- Design a stabilizing controller $u := Kx$ such that $\lim_{t\to\infty} \|x(t)\| \to 0$ in the Lyapunov sense.
- Let $V(x) := x^T P x, P = P^T > 0$ be a candidate Lyapunov function
- Therefore,

$$\begin{aligned}\dot{V} &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T \left((A+BK)^T P + P(A+BK)\right) x.\end{aligned}$$

$\dot{V} \leq 0 \implies$

$$A^T P + PA + K^T B^T P + PBK \leq 0.$$

BMI in P, K.

Stabilizing Controller

contd.

BMI in P, K:

$$A^T P + PA + K^T B^T P + PBK \leq 0.$$

Use substitution

$$P := Y^{-1},\ K := WY^{-1}$$

$$A^T Y^{-1} + Y^{-1} A + Y^{-1} W^T B^T Y^{-1} + Y^{-1} BWY^{-1} \leq 0.$$

Multiply both sides by Y, congruent transformation

LMI in Y, W

$$YA^T + AY + W^T B^T + BW \leq 0.$$

Stabilizing Controller

Bounded Exponent

Lemma

$$\dot{V} \leq -\alpha V \implies \|x(t)\|_2^2 \leq \beta \|x(0)\|_2^2 \, e^{-\alpha(t-t_0)}$$

Proof:

$$\begin{aligned} \dot{V} &\leq -\alpha V \\ \frac{dV}{V} &\leq -\alpha d\tau. \end{aligned}$$

Integrating from $[t_0, t]$, we get

$$\begin{aligned} V(x(t)) &\leq V(x(0))e^{\alpha(t-t_0)} \\ x^T P x &\leq x(0)^T P x(0) e^{\alpha(t-t_0)}. \end{aligned}$$

Stabilizing Controller

Bounded Exponent (contd.)

Recall

$$\lambda_{\mathsf{min}}(P)\|x\|_2^2 \leq x^T P x \leq \lambda_{\mathsf{max}}(P)\|x\|_2^2$$

Implies

$$\lambda_{\mathsf{min}}(P)\|x\|_2^2 \leq \lambda_{\mathsf{max}}(P)\|x(0)\|_2^2 e^{-\alpha(t-t_0)}$$

or

$$\|x\|_2^2 \leq \kappa(P)\|x(0)\|_2^2 e^{-\alpha(t-t_0)}.$$

The condition

$$\dot{V} \leq -\alpha V$$

for dynamical system $\dot{x} = (A + BK)x$ results in the following LMI in Y, W

LMI for Bounded Exponent

$$YA^T + AY + W^T B^T + BW + \alpha Y \leq 0.$$

Stabilizing Controller

Bounded Exponent (contd.)

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Stabilizing with Finsler's Lemma

Finsler's Lemma

Lemma (Finsler) Consider $w \in \mathbb{R}^{n_x}, \mathcal{L} \in \mathbb{R}^{n_x \times n_x}$, and $\mathcal{B} \in \mathbb{R}^{m_x \times n_x}$ with rank$(\mathcal{B}) < n_x$, and $\mathcal{B}^\perp$ a basis for the null space of $\mathcal{B}$ ($\mathcal{B}\mathcal{B}^\perp = 0$). The following conditions are equivalent:

1. $w^T \mathcal{L} w < 0,\ \forall w \neq 0 : \mathcal{B}w = 0$
2. ${\mathcal{B}^\perp}^T \mathcal{L} \mathcal{B}^\perp < 0$
3. $\exists \mu \in \mathbb{R} : \mathcal{L} - \mu \mathcal{B}^T \mathcal{B} < 0$
4. $\exists \mathcal{X} \in \mathbb{R}^{n_x \times m_x} : \mathcal{L} + \mathcal{X}\mathcal{B} + \mathcal{B}^T \mathcal{X}^T < 0$

Proof: Olivera & Skelton, 2001.

Finsler's Lemma

contd.

Given closed-loop system $\dot{x} = (A + BK)x$, define the following

$$w := \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

$$\mathcal{B} := \begin{bmatrix} (A+BK) & -I \end{bmatrix}, \implies \mathcal{B}^{\perp} = \begin{bmatrix} I \\ (A+BK) \end{bmatrix},$$

$$\mathcal{L} := \begin{bmatrix} \alpha P & P \\ P & 0 \end{bmatrix}.$$

Property 1 of Finsler's Lemma:

$$\mathcal{B}w = 0 \iff \dot{x} = (A+BK)x$$

$$w^T \mathcal{L} w < 0 \iff x^T \left((A+BK)^T P + P(A+BK) + \alpha P \right) x < 0$$

Finsler's Lemma

contd.

Property 2 of Finsler's Lemma: $\exists P = P^T > 0$ such that

$$\begin{bmatrix} I \\ (A+BK) \end{bmatrix}^T \begin{bmatrix} \alpha P & P \\ P & 0 \end{bmatrix} \begin{bmatrix} I \\ (A+BK) \end{bmatrix} < 0,$$

which is equivalent to

$$(A+BK)^T P + P(A+BK) + \alpha P < 0.$$

Property 3 of Finsler's Lemma:

$$\mathcal{L} - \mu \mathcal{B}^T \mathcal{B} < 0 \iff \begin{bmatrix} \alpha P - \mu (A+BK)^T (A+BK) & P + \mu (A+BK)^T \\ P + \mu (A+BK) & -\mu I \end{bmatrix} < 0.$$

Finsler's Lemma

contd.

Schur complement of

$$\begin{bmatrix} \alpha P - \mu(A+BK)^T(A+BK) & P + \mu(A+BK)^T \\ P + \mu(A+BK) & -\mu I \end{bmatrix} < 0$$

implies $-\mu I < 0$, and

$$\gamma P - \mu(A+BK)^T(A+BK) - \left(P + \mu(A+BK)^T\right)(-\mu I)^{-1}\left(P + \mu(A+BK)^T\right) < 0$$

$$\implies (A+BK)^T P + P(A+BK) + \alpha P < \frac{PP}{\mu} \text{ trivial}$$

Finsler's Lemma

contd.

Property 4 of Finsler's Lemma: $\exists \mathcal{X} \in \mathbb{R}^{n_x \times m_x}$ such that

$$\mathcal{L} + \mathcal{X}\mathcal{B} + \mathcal{B}^T\mathcal{X}^T < 0.$$

or

$$\begin{bmatrix} \gamma P & P \\ P & 0 \end{bmatrix} + \mathcal{X}\begin{bmatrix} (A+BK) & -I \end{bmatrix} + \begin{bmatrix} (A+BK)^T \\ -I \end{bmatrix}\mathcal{X}^T < 0.$$

Define $\mathcal{X} := \begin{bmatrix} Z \\ aZ \end{bmatrix}$, with $Z \in \mathbb{R}^{n\times n}$, invertible but not necessarily symmetric, and $a > 0$ a fixed relaxation constant.

Finsler's Lemma

contd.

Substituting and applying congruent transformation $\begin{bmatrix} Z^{-1} & 0 \\ 0 & Z^{-1} \end{bmatrix}$ on left and $\begin{bmatrix} Z^{-1} & 0 \\ 0 & Z^{-1} \end{bmatrix}^T$ on the right we get

$$\begin{bmatrix} Z^{-1}(A^T + K^T B^T) + (*)^T + \alpha Z^{-1} P Z^{-T} & (*)^T \\ Z^{-1} P Z^{-T} + a(A + BK) Z^{-T} - Z^{-1} & -a(Z^{-1} + Z^{-T}) \end{bmatrix} < 0$$

Finsler's Lemma

contd.

$$\begin{bmatrix} Z^{-1}(A^T + K^T B^T) + (*)^T + \alpha Z^{-1} P Z^{-T} & (*)^T \\ Z^{-1} P Z^{-T} + a(A+BK)Z^{-T} - Z^{-1} & -a(Z^{-1} + Z^{-T}) \end{bmatrix} < 0$$

Substitute:

$$Y := Z^{-T}, \qquad W := KY, \qquad \text{and} \quad Q := Y^T P Y$$

we get

$$\begin{bmatrix} AY + Y^T A^T + BW + W^T B^T + \alpha Q & Q + a(Y^T A^T + W^T B^T) - Y \\ Q + a(AY + BW) - Y^T & -a(Y + Y^T) \end{bmatrix} < 0.$$

Variables: $Y \in \mathbb{R}^{n\times n} \neq Y^T, W \in \mathbb{R}^{m\times n}$ and $Q = Q^T \in \mathbb{R}^{n\times n} > 0$. Parameter a is given.

Greater degree of freedom

Stabilizing with Reciprocal Projection Lemma

Reciprocal Projection Lemma

Recall With $X = X^T > 0$, the following are true

$$\Psi + S + S^T < 0 \iff \begin{bmatrix} \Psi + X - (W + W^T) & S^T + W^T \\ S + W & -X \end{bmatrix} < 0.$$

Consider Lyapunov inequality with decay-rate

$$(A + BK)Y + Y(A + BK)^T + \alpha Y < 0, \; Y > 0 \quad V(x) := x^T Y^{-1} x$$

Let

$$\Psi := 0, S^T := (A + BK)Y + \frac{\alpha}{2}Y$$

Implies

$$\Psi + S^T + S < 0 \iff (A + BK)Y + Y(A + BK)^T + \alpha Y < 0.$$

Reciprocal Projection Lemma

contd.

From Reciprocal Projection Lemma we get

$$\Psi + S^T + S < 0 \iff (A+BK)Y + Y(A+BK)^T + \alpha Y < 0$$

$$\iff \begin{bmatrix} X - (W + W^T) & (A+BK)Y + \frac{\alpha}{2}Y + W^T \\ Y(A+BK)^T + \frac{\alpha}{2}Y + W & -X \end{bmatrix} < 0.$$

Multiplying on both sides by $\begin{bmatrix} I & 0 \\ 0 & Y^{-1} \end{bmatrix}$ we get

$$\begin{bmatrix} X - (W + W^T) & (A+BK) + \frac{\alpha}{2}I + W^T P \\ (A+BK)^T + \frac{\alpha}{2}I + PW & -PXP \end{bmatrix} < 0$$

Reciprocal Projection Lemma

contd.

Multiply

$$\begin{bmatrix} X-(W+W^T) & (A+BK)+\frac{\alpha}{2}I+W^TP \\ (A+BK)^T+\frac{\alpha}{2}I+PW & -PXP \end{bmatrix} < 0$$

on left hand side with $\begin{bmatrix} W^{-1} & 0 \\ 0 & I \end{bmatrix}^T$ and right hand side with $\begin{bmatrix} W^{-1} & 0 \\ 0 & I \end{bmatrix}$, and substitute $V := W^{-1}$ to get

$$\begin{bmatrix} V^TXV-(V+V^T) & V^T(A+BK)+\frac{\alpha}{2}V^T+P \\ (A+BK)^TV+\frac{\alpha}{2}V+P & -PXP \end{bmatrix} < 0.$$

Reciprocal Projection Lemma

contd.

Using Schur complement it can be shown that

$$\begin{bmatrix} V^TXV - (V+V^T) & V^T(A+BK) + \frac{\alpha}{2}V^T + P \\ (A+BK)^TV + \frac{\alpha}{2}V + P & -PXP \end{bmatrix} < 0.$$

is equivalent to

$$\begin{bmatrix} -(V+V^T) & V^T(A+BK) + \frac{\alpha}{2}V^T + P & V^T \\ (A+BK)^TV + \frac{\alpha}{2}V + P & -PXP & 0 \\ V & 0 & -X^{-1} \end{bmatrix} < 0.$$

Now substitute $X := P^{-1}$ to get

$$\begin{bmatrix} -(V+V^T) & V^T(A+BK) + \frac{\alpha}{2}V^T + P & V^T \\ (A+BK)^TV + \frac{\alpha}{2}V + P & -P & 0 \\ V & 0 & -P \end{bmatrix} < 0.$$

Reciprocal Projection Lemma

contd.

Using the dual form $(A+BK) \mapsto (A+BK)^T$ we get (Apkarian et.al 2001)

$$\begin{bmatrix} -(V+V^T) & V^T(A+BK)^T + \frac{\alpha}{2}V^T + P & V^T \\ (A+BK)V + \frac{\alpha}{2}V + P & -P & 0 \\ V & 0 & -P \end{bmatrix} < 0.$$

With change of variable $Z := KV$, we get the final LMI

$$\begin{bmatrix} -(V+V^T) & V^TA^T + Z^TB^T + \frac{\alpha}{2}V^T + P & V^T \\ AV + BZ + \frac{\alpha}{2}V + P & -P & 0 \\ V & 0 & -P \end{bmatrix} < 0.$$

Variables $P > 0 \in \mathbb{R}^{n\times n}, V \in \mathbb{R}^{n\times n}, Z \in \mathbb{R}^{m\times n}$.

Controller is decoupled from Lyapunov function $K := ZV^{-1}$

Minimum Norm Controller

Pointwise minimum norm

We are interested in stabilizing controller that minimizes instantaneous $u^T u$, where

$$u^T u = x^T K^T K x = x^T Y^{-1} W^T W Y^{-1} x.$$

Optimization problem 1

$$\begin{aligned} &\min \gamma \\ Y &\geq \mu_0 I_n \\ W^T W &\leq \gamma I_m, \\ YA^T + AY + W^T B^T + BW + \alpha Y &\leq 0. \end{aligned}$$

or

$$\min \gamma, \text{ subject to } \begin{array}{c} YA^T + AY + W^T B^T + BW + \alpha Y \leq 0 \\ \begin{bmatrix} \gamma I_m & W \\ W^T & I_n \end{bmatrix} \geq 0, Y \geq \mu_0 I_n. \end{array}$$

Minimum Gain

Better Formulation

$$
\begin{aligned}
&\min \gamma, \\
&\begin{bmatrix} Y & W^T \\ W & \gamma I_m \end{bmatrix} \geq 0, \\
&Y \geq \mu_0 I_n, \\
&YA^T + AY + W^T B^T + BW + \alpha Y \leq 0.
\end{aligned}
$$

From Schur complement about γI_mwe get

$$\gamma I_m > 0,\ Y - W^T (\gamma I_m)^{-1} W \geq 0.$$

Or

$$W^T W \leq \gamma Y.$$

Minimum Gain

Better Formulation (contd.)

We have

$$W^T W \leq \gamma Y.$$

Substitute $W = KY$, to get

$$YK^T KY \leq \gamma Y \implies K^T K \leq \gamma Y^{-1}.$$

But $X \geq \mu_0 I_n$ constraint in LMI.
Therefore

$$K^T K \leq \frac{\gamma}{\mu_0} I_n.$$

Minimum Gain

With Finsler's Lemma

$$\min \gamma,$$

$$\begin{bmatrix} Q & W^T \\ W & \gamma I_m \end{bmatrix} \geq 0,$$

$$Q \geq \mu_0 I_n,$$

$$\begin{bmatrix} AY + Y^T A^T + BW + W^T B^T + \alpha Q & Q + a(Y^T A^T + W^T B^T) - Y \\ Q + a(AY + BW) - Y^T & -a(Y + Y^T) \end{bmatrix} < 0.$$

Minimum Gain

With Reciprocal Projection Lemma

$$\min \gamma,$$

$$\begin{bmatrix} I_n & Z^T \\ Z & \gamma I_m \end{bmatrix} > 0$$

$$P \geq \mu_0 I_n,$$

$$\begin{bmatrix} -(V + V^T) & V^T A^T + Z^T B^T + \frac{\alpha}{2} V^T + P & V^T \\ AV + BZ + \frac{\alpha}{2} V + P & -P & 0 \\ V & 0 & -P \end{bmatrix} < 0.$$

Linear Quadratic Regulator

Synthesis Problem

Find $P = P^T > 0$ and K such that with $V := x^T P x$,

$$\min_{P,K} V(x(0)) = x(0)^T P x(0) \text{ Cost Function}$$

subject to

$$\dot{V} \leq -x^T (Q + K^T R K) x \text{ Constraint Function}$$

Or equivalently

$$\min_{P,K} \mathbf{tr} P$$

subject to

$$(A + BK)^T P + P(A + BK) + Q + K^T R K \leq 0.$$

Synthesis Problem

contd.

$$\min_{P,K} \mathbf{tr} P$$

subject to

$$(A+BK)^T P + P(A+BK) + Q + K^T R K \leq 0$$

is not an LMI.

Substitution: $Y := P^{-1}$, and $W := KY$ and applying congruent transformation with Y we get

$$AY + YA^T + W^T B^T + BW + YQY + W^T R W \leq 0.$$

Synthesis Problem

contd.

Matrix inequality

$$AY + YA^T + W^TB^T + BW + YQY + W^TRW \leq 0,$$

is equivalent to

$$\begin{bmatrix} AY + YA^T + W^TB^T + BW & Y & W^T \\ Y & -Q^{-1} & 0 \\ W & 0 & -R^{-1} \end{bmatrix} \leq 0.$$

or

$$\begin{bmatrix} AY + YA^T + W^TB^T + BW & (\sqrt{Q}Y)^T & (\sqrt{R}W)^T \\ \sqrt{Q}Y & -I_n & 0 \\ \sqrt{R}W & 0 & -I_m \end{bmatrix} \leq 0.$$

Synthesis Problem

contd.

Therefore, synthesis optimization problem is

$$\max_{Y,W} \mathbf{tr} Y$$

subject to

$$\begin{bmatrix} AY + YA^T + W^TB^T + BW & (\sqrt{Q}Y)^T & (\sqrt{R}W)^T \\ \sqrt{Q}Y & -I_n & 0 \\ \sqrt{R}W & 0 & -I_m \end{bmatrix} \leq 0.$$

The solution is the same as Riccati solution

Synthesis Problem

Solution

If $K = -R^{-1}B^TY^{-1} = WY^{-1}$, then

$$W = -R^{-1}B^T.$$

Substitute it in

$$AY + YA^T + W^TB^T + BW + YQY + W^TRW \leq 0.$$

to get

$$AY + YA^T - BR^{-1}B^T + YQY \leq 0,$$

or

$$AY + YA^T + YQY \leq \underbrace{BR^{-1}B^T}_{\geq 0}.$$

$$\max \mathbf{tr} Y \implies AY + YA^T + YQY - BR^{-1}B^T = 0.$$ Max at boundary

With $K = -R^{-1}B^TP$

With the controller $K = -R^{-1}B^TP$, the condition

$$(A + BK)^TP + P(A + BK) + Q + K^TRK \leq 0,$$

becomes

$$A^TP + PA + Q - PBR^{-1}B^TP \leq 0.$$

This is not a convex constraint in P

Schur complement (about A_{22}) of

$$\begin{bmatrix} A^TP + PA + Q & PB \\ B^TP & R \end{bmatrix} \leq 0.$$

gives

$$A^TP + PA + Q - PBR^{-1}B^TP \leq 0, R \leq 0.$$

With $K = -R^{-1}B^TP$

contd.

Let $Y := P^{-1}$ and substitute in

$$A^TP + PA + Q - PBR^{-1}B^TP \leq 0,$$

to get

$$A^T Y^{-1} + Y^{-1}A + Q - Y^{-1}BR^{-1}B^TY^{-1} \leq 0.$$

Multiply by Y on both sides congruent transform

$$YA^T + AY + YQY - BR^{-1}B^T \leq 0.$$

This is convex

$$\begin{bmatrix} YA^T + AY - BR^{-1}B^T & Y\sqrt{Q} \\ \sqrt{Q}Y & -I \end{bmatrix} \leq 0.$$

With $K = -R^{-1}B^TP$

contd.

Therefore optimization problem is

$$\max_{Y} \mathbf{tr} Y$$

subject to

$$\begin{bmatrix} YA^T + AY - BR^{-1}B^T & Y\sqrt{Q} \\ \sqrt{Q}Y & -I_n \end{bmatrix} \leq 0.$$

Minimization of $\|y\|_2^2$

System is

$$\dot{x} = Ax + Bu, \; y = Cx + Du.$$

Then

$$\|y\|_2^2 = \int_0^\infty (x^T C^T C x + u^T D^T D u) dt,$$

assuming (for simplicity) $D^T D$ is invertible and $D^T C = 0$.

Therefore, with

$$Q = C^T C, \; R = D^T D,$$

Optimization problem in Y is

$$\max_Y \mathbf{tr} Y, \text{ subject to } \begin{bmatrix} YA^T + AY - B(D^T D)^{-1} B^T & YC^T \\ CY & -I_n \end{bmatrix} \leq 0.$$

How does it relate to Riccati Solution

Apply "completion of squares" idea.

Consider

$$\begin{aligned}\dot{V} &= \frac{d}{dt}x^TPx \\ &= x^T(A^TP + PA)x + x^TPBu + u^TB^TPx.\end{aligned}$$

Add $x^TQx + u^TRu$ on both sides to get

$$\dot{V} + x^TQx + u^TRu = x^T(A^TP + PA)x + x^TPBu + u^TB^TPx \\ + x^TQx + u^TRu.$$

- Add and subtract $x^TPBR^{-1}B^TPx$ on RHS.
- Let $R = U^TU$ for some square invertible U.

How does it relate to Riccati Solution

contd.

Therefore

$$\dot{V} + x^T Q x + u^T R u = x^T (A^T P + PA)x + x^T PBu + u^T B^T Px + x^T Qx + u^T Ru - x^T PBR^{-1} B^T Px + x^T PBR^{-1} B^T Px.$$

Or

$$\dot{V} + x^T Q x + u^T R u = x^T \left(A^T P + PA + Q - PBR^{-1} B^T P\right) x + \|Uu + U^{-T} B^T Px\|^2.$$

How does it relate to Riccati Solution

contd.

$$\dot{V} + x^T Qx + u^T Ru = x^T \left(A^T P + PA + Q - PBR^{-1}B^T P\right) x + \|Uu + U^{-T}B^T Px\|^2.$$

Let P be such that

$$A^T P + PA + Q - PBR^{-1}B^T P = 0.$$

Then,

$$\dot{V} + x^T Qx + u^T Ru = \|Uu + U^{-T}B^T Px\|^2.$$

How does it relate to Riccati Solution

contd.

Integrating

$$\dot{V} + x^T Q x + u^T R u = \|Uu + U^{-T} B^T P x\|^2 \geq 0$$

over $[0, T]$ we get

$$x(T)^T P x(T) + \int_0^T (x^T Q x + u^T R u) dt \geq x_0^T P x_0$$

With $T \to \infty$, $x(T) \to 0$

$$\implies \int_0^T (x^T Q x + u^T R u) dt \geq x_0^T P x_0 \text{ Lower Bound}$$

Equality when

$$Uu + U^{-T} B^T P x = 0 \implies u = -R^{-1} B^T P x.$$